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An R-matrix approach to the quantization of the Euclidean group $E(2)$

A Ballesteros†§, E Celeghini†, R Giachetti‡‖, E Sorace† and M Tarlini†

† Dipartimento di Fisica, Università di Firenze and INFN-Firenze, Italy

‡ Dipartimento di Matematica, Università di Bologna and INFN-Firenze, Italy

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Abstract. The R-matrices for two different deformations of the Euclidean group $E(2)$, calculated in a two-dimensional representation, are used to determine the deformed Hopf algebra of the representative functions. The duality of the latter with the initial quantum algebras is explicitly proved and the relationship between the two quantum groups is discussed and clarified.

1. Introduction

The quantization of compact Lie groups has reached a considerable level of completeness, both on the side of the algebraic structure as well as on the determination and classification of the representations. The state of the art is different for non-semisimple groups. From the point of universal enveloping algebras the lack of a root decomposition prevents a well defined deformation procedure; from the point of the algebra of the representative functions difficulties are encountered due to the absence of general results like the Peter–Weyl theorem or the Tannaka duality (see [1] for the quantum version of these results).

One of the simplest non-compact non-semisimple groups that has undergone a quantization procedure is the two-dimensional Euclidean group. The problem has been studied at a von Neumann algebra level in [2], where the q-analogue of Bessel functions has been introduced. A purely algebraic treatment has been considered in [3] in connection with a general mechanism of contraction of Hopf algebras (see also [4]). Besides the quantum structure given in [2], a new deformation has been found in [3], presenting the interesting feature that the quantum parameter can be reabsorbed by a suitable rescaling of the generators. In [5] the direct relevance of the pseudo-Euclidean version of this quantum algebra to phonon physics has been presented. A C^*-algebra approach has finally been discussed in [6, 7] and the analogue of the algebra of continuous functions vanishing at infinity has been determined.

In some cases the contraction mechanism has also allowed for the calculation of an R matrix: whenever this has been possible, as for $H_q(1)$ and $E_q(3)$ [3, 4], the quantization of the algebra of the representative functions has been obtained by using the fundamental representation in the framework proposed in [8], although it has not been shown in general that the dual of the initial Hopf algebra is actually obtained. Unfortunately this circumstance does not occur for the two-dimensional Euclidean group: the contraction of the R matrix is badly divergent and the universal R matrix does not exist. A direct contraction of the

§ On leave from: Departamento de Física Teórica, Universidad de Valladolid, 47011-Valladolid, Spain.

‖ E-mail address: GIACHETTI@FI.INFN.IT .

quantum group structure has been proposed in [7] by means of an approach leaning on two-dimensional matrix representations of $SU(2)$ and $E(2)$: the results are in full agreement with [2], as has been shown in [9].

Despite the impossibility of getting an algebraic expression, in this paper we show that a direct calculation of the R matrix for the two deformations of $E(2)$ can be made in a two-dimensional representation substantially equivalent to that presented in [7]. The consequent quantizations of the group in the scheme proposed in [8] can then be studied. What is found is that, in both cases, this procedure is not able to specify all the relations between the generators of $\mathrm{Fun}(E_q(2))$: the R matrix provides only some of them, the remaining ones being determined by internal consistency and by the compatibility with the coproduct. The actual calculations show that the quantum group given in [2,7] is recovered and that a new quantum group without parameters is found: it is then shown that this quantum group is exactly the dual Hopf algebra to that given in [3]. We observe that an analogous calculation in a three-dimensional representation cannot be developed, since in such a representation no R matrix verifying the quantum Yang–Baxter equation exists.

To conclude, we push the investigation a little further and we see whether the consistency conditions and the compatibility with the coproduct alone are able to produce a slightly more general structure. In fact, using the two-dimensional representation, we find the most general expression for the product of the quantum group generators which are depending upon two deformation parameters. Again, one of these parameters can be eliminated by rescaling the generators. It is then realized that, when the other is non-vanishing, we can make a change of basis which shows that the structure we thus get is that given in [2,7]. However, for vanishing value of that parameter the transformation has a singularity and cannot be made: in this case the quantum group we obtain is just the quantum group without parameters. Thus we can draw an exhaustive picture of the deformations of $E(2)$ and of the relationships between them.

2. The R-matrix way to Euclidean quantum groups

Let us first briefly recall the two contractions of $SU_q(2)$, leading to different quantizations of the universal enveloping algebra of $E(2)$. The first, [2], is obtained by rescaling the $SU_q(2)$ generators as follows:

$$^t(P_x, P_y, J, z) = \mathrm{diag}\,\{\varepsilon, \varepsilon, 1, 1\}\,^t(J_1, J_2, J_3, z).$$

Taking the limit $\varepsilon \to 0$ we obtain a quantum algebra whose relations are

$$[P_x, P_y] = 0 \qquad [J, P_x] = \mathrm{i}P_y \qquad [J, P_y] = -\mathrm{i}P_x \tag{2.1}$$

where $\mathrm{e}^z = q$. The coproducts, antipodes and co-units are

$$\begin{aligned} &\Delta J = 1 \otimes J + J \otimes 1 \qquad \Delta P_i = \mathrm{e}^{-zJ/2} \otimes P_i + P_i \otimes \mathrm{e}^{zJ/2} \\ &\gamma(J) = -J \qquad \gamma(P_i) = -\mathrm{e}^{zJ/2} P_i \mathrm{e}^{-zJ/2} \\ &\epsilon(P_i) = \epsilon(J) = 0 \end{aligned} \tag{2.2}$$

where $i = x, y$. We shall refer to it as $E_q(2)$.

The second quantization of the two-dimensional Euclidean algebra comes from the rescaling

$$ {}^{t}(J, P_y, P_x, w) = \mathrm{diag}\{1, \varepsilon, \varepsilon, \varepsilon^{-1}\}\, {}^{t}(J_1, J_2, J_3, z) $$

thus defining a new parameter w. The quantum algebra resulting in the limit $\varepsilon \to 0$ satisfies the relations

$$ [P_x, P_y] = 0 \qquad [J, P_x] = \mathrm{i}P_y \qquad [J, P_y] = -(\mathrm{i}/w)\sinh(wP_x)\,. \tag{2.3} $$

Moreover

$$ \begin{aligned} &\Delta P_x = 1 \otimes P_x + P_x \otimes 1 \qquad \Delta P_y = \mathrm{e}^{-wP_x/2} \otimes P_y + P_y \otimes \mathrm{e}^{wP_x/2} \\ &\Delta J = \mathrm{e}^{-wP_x/2} \otimes J + J \otimes \mathrm{e}^{wP_x/2} \\ &\gamma(P_x) = -P_x \qquad \gamma(P_y) = -P_y \qquad \gamma(J) = -J + \tfrac{1}{2}\mathrm{i}wP_y \\ &\epsilon(P_x) = \epsilon(P_y) = \epsilon(J) = 0\,. \end{aligned} \tag{2.4} $$

It is then seen that, letting $P_i \longrightarrow P_i/w$, the deformation parameter can be reabsorbed. Due to its relevance in lattice physics [5], we shall refer to it as $E_\ell(2)$ (see [3] for details).

Let us now consider the algebra $\mathrm{Fun}(E(2))$ obtained by a two-dimensional representation with matrices of the form

$$ T = \begin{pmatrix} v & n \\ 0 & 1 \end{pmatrix} \tag{2.5} $$

with v unitary and n complex. The coproducts of the matrix elements are

$$ \Delta v = v \otimes v \qquad \Delta n = v \otimes n + n \otimes 1 \qquad \Delta 1 = 1 \otimes 1\,. \tag{2.6} $$

Making the extension by the conjugate $\bar{v}$ of the group-like unitary generator v, $\bar{v}v = v\bar{v} = 1$, and by the conjugate $\bar{n}$ of n, with

$$ \Delta\bar{v} = \bar{v} \otimes \bar{v} \qquad \Delta\bar{n} = \bar{v} \otimes \bar{n} + \bar{n} \otimes 1 \tag{2.7} $$

we have the antipodes

$$ \begin{aligned} &\gamma(v) = \bar{v} \qquad \gamma(\bar{v}) = v \qquad \gamma(1) = 1 \\ &\gamma(n) = -\bar{v}n \qquad \gamma(\bar{n}) = -v\bar{n} \end{aligned} \tag{2.8} $$

and co-units

$$ \epsilon\,(v) = \epsilon\,(\bar{v}) = \epsilon\,(1) = 1 \qquad \epsilon\,(n) = \epsilon\,(\bar{n}) = 0\,. \tag{2.9} $$

The quantization of $\mathrm{Fun}(E(2))$ is obtained by establishing non-commutative deformed relations among the generators, compatible with (2.6)–(2.9). This is the general framework used when a universal R matrix exists. Since for the two-dimensional Euclidean quantum group there is no universal R matrix we shall determine such relations by calculating the R matrices for the quantum structures $E_q(2)$ and $E_\ell(2)$ directly in the representation (2.5).

To this purpose we observe that the infinitesimal generators, both in the quantum and non-quantum cases, are represented by

$$P_x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad P_y = \begin{pmatrix} 0 & \mathrm{i} \\ 0 & 0 \end{pmatrix} \qquad J = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{2.10}$$

In the first place we shall look for a 4×4 matrix which transposes the coproduct, i.e. which satisfies the relation

$$R \Delta R^{-1} = P \circ \Delta$$

where P is the operator interchanging the two factors of the tensor product. Secondly this matrix is required to solve the Yang–Baxter equation

$$R_{12}\, R_{13}\, R_{23} = R_{23}\, R_{13}\, R_{12}\,.$$

Besides the trivial solution $R = P$, for the quantum algebra $E_q(2)$, we find, up to an equivalence, a unique solution R_q depending on the deformation parameter, namely

$$R_q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 - \mathrm{e}^{-z} & \mathrm{e}^{-z} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.11}$$

It must be observed that the R matrix (2.11) is a special case of the general R matrix coming from the two-parameter deformation of $GL(2)$, [10, 11]. Indeed, in the notation of [11] and up to a trivial change of basis, the expression (2.11) is just $R_{p,q}$ with $p = 1$ and $q = \mathrm{e}^{-z}$.

Using the well known prescription

$$R\, T_1\, T_2 = T_2\, T_1\, R \tag{2.12}$$

with $T_1 = T \otimes 1$, $T_2 = 1 \otimes T$, we find the following relations between the generators of the algebra $\mathrm{Fun}(E_q(2))$:

$$vn = \mathrm{e}^{z}\, nv \qquad n\bar{v} = \mathrm{e}^{z}\, \bar{v}n \qquad v\bar{v} = 1\,. \tag{2.13}$$

Assuming real z and conjugating (2.13) we then find

$$\bar{n}\bar{v} = \mathrm{e}^{z}\, \bar{v}\bar{n} \qquad v\bar{n} = \mathrm{e}^{z}\, \bar{n}v\,. \tag{2.14}$$

However, no relation between n and $\bar{n}$ can be obtained from (2.12). We can overcome this difficulty by requiring compatibility with (2.13) and (2.14) and with the homomorphism property of the coproduct. We get the solution

$$n\bar{n} = \mathrm{e}^{z}\bar{n}n$$

showing that v, n and $\bar{n}$ generate the algebra $\mathrm{Fun}(E_q(2))$ given in [7]. More exactly, if instead of (2.5) we used the representation

$$\tilde{T} = \begin{pmatrix} u & c \\ 0 & \bar{u} \end{pmatrix}$$

we would find the R matrix

$$\tilde{R}_q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{-z/2} & 0 & 0 \\ 0 & 1 - e^{-z/2} & e^{-z/2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

just giving the results found in [7] and equivalent to (2.13) and (2.14) provided we make the identifications [9]

$$v = u^2 \qquad n = cu\,. \tag{2.15}$$

The very same procedure applied to $E_\ell(2)$ yields the R matrix

$$R_\ell = \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.16}$$

which, again, is a special case of the two-parameter R matrix referred to in [11] as the 'Jordanian solution'. From (2.16) we obtain the equation

$$vn - nv = v^2 - v \tag{2.17}$$

the conjugate one and those deduced by the use of $v\bar{v} = \bar{v}v = 1$. For the relation involving n and $\bar{n}$ we follow the previous approach obtaining

$$[n, \bar{n}] = -(n + \bar{n}) \tag{2.18}$$

which completes the new Hopf algebra $\mathrm{Fun}(E_\ell(2))$.

3. Duality

The duality of $E_q(2)$ with $\mathrm{Fun}(E_q(2))$ has been explicitly proved in [9] and the topological aspects related to $\mathrm{Fun}(E_q(2))$ have been deeply investigated in [6, 7]. Neglecting a detailed topological study, in this section we shall prove that an analogous algebraic duality holds between $E_\ell(2)$ and $\mathrm{Fun}(E_\ell(2))$.

We find it convenient to introduce the generators t, n_1, n_2, by means of the definitions

$$v = e^{it} \qquad n_1 = \tfrac{1}{2}(n + \bar{n}) \qquad n_2 = \tfrac{1}{2}i(n - \bar{n}) \tag{3.1}$$

so that $\mathrm{Fun}(E_\ell(2))$ will be spanned by the ordered monomials in t, n_1, n_2. A straightforward calculation yields the commutators

$$[t, n_1] = i\,(1 - \cos t) \qquad [t, n_2] = i\,\sin t \qquad [n_1, n_2] = i\,n_1\,. \tag{3.2}$$

Let us introduce the elements τ, ν_1, ν_2, dual to t, n_1, n_2, i.e. satisfying

$$\begin{aligned} &\langle \tau, t^a n_1^b n_2^c \rangle = \delta_{a1}\delta_{b0}\delta_{c0} \qquad \langle \nu_1, t^a n_1^b n_2^c \rangle = \delta_{a0}\delta_{b1}\delta_{c0} \\ &\langle \nu_2, t^a n_1^b n_2^c \rangle = \delta_{a0}\delta_{b0}\delta_{c1}\,. \end{aligned} \tag{3.3}$$

We shall first use the multiplication in $\mathrm{Fun}(E_\ell(2))$ in order to determine the coproduct of the dual $\mathrm{Fun}(E_\ell(2))'$ according to

$$\Delta(\alpha)(a\otimes b) = \langle \alpha, ab\rangle \qquad \alpha\in \mathrm{Fun}(E_\ell(2))' \qquad a,b\in \mathrm{Fun}(E_\ell(2))\,.$$

Using (3.2) and (3.3), we then find

$$\begin{aligned} &\Delta(\tau) = \tau\otimes 1 + \mathrm{e}^{-\mathrm{i}\nu_2}\otimes\tau \qquad \Delta(\nu_1) = \nu_1\otimes 1 + \mathrm{e}^{-\mathrm{i}\nu_2}\otimes\nu_1 \\ &\Delta(\nu_2) = \nu_2\otimes 1 + 1\otimes\nu_2\,. \end{aligned} \tag{3.4}$$

Conversely, let us consider the coproduct in $\mathrm{Fun}(E_\ell(2))$ for obtaining the multiplication—and hence the commutation relations—in $\mathrm{Fun}(E_\ell(2))'$. As a result we have

$$[\tau,\nu_1] = -\tfrac{1}{2}\mathrm{i}\,\nu_1^2 + \tfrac{1}{2}\mathrm{i}\big(1-\mathrm{e}^{-2\mathrm{i}\nu_2}\big) \qquad [\tau,\nu_2] = \nu_1 \qquad [\nu_1,\nu_2] = 0\,. \tag{3.5}$$

In order to obtain the antipode γ in $\mathrm{Fun}(E_\ell(2))'$ we use the relation $\langle\gamma(\alpha),a\rangle = \langle a,\gamma(\alpha)\rangle$ and we get

$$\gamma(\tau) = -\mathrm{e}^{\mathrm{i}\nu_2}\tau \qquad \gamma(\nu_1) = -\mathrm{e}^{\mathrm{i}\nu_2}\nu_1 \qquad \gamma(\nu_2) = -\nu_2\,. \tag{3.6}$$

Finally we can determine an involution $\alpha\mapsto\alpha^*$ in $\mathrm{Fun}(E_\ell(2))'$ by $\langle\alpha^*,a\rangle = \overline{\langle\alpha,\gamma^{-1}(\bar a)\rangle}$. The result is

$$\tau^* = -\tau + \mathrm{i}\nu_1 \qquad \nu_1^* = -\nu_1 \qquad \nu_2^* = -\nu_2\,. \tag{3.7}$$

We are now able to compare $\mathrm{Fun}(E_\ell(2))'$ with our initial algebra $E_\ell(2)$. The mapping

$$\tau = \mathrm{i}\mathrm{e}^{-P_x/2}(J+\tfrac{1}{4}\mathrm{i}P_y) \qquad \nu_1 = \mathrm{i}\mathrm{e}^{-P_x/2}P_y \qquad \nu_2 = -\mathrm{i}P_x \tag{3.8}$$

is compatible with $J^*=J$, $P_x^*=P_x$, $P_y^*=P_y$ and a comparison of (3.4)–(3.8) with (2.3), (2.4) shows that indeed $\mathrm{Fun}(E_\ell(2))'$ coincides with $E_\ell(2)$.

We shall conclude this paper by considering the relationship between the two deformations of $E(2)$ we have so far discussed. We now consider again the representation (2.5) with the related formulae (2.4)–(2.9) and we look for general expressions of the products vn and $n\bar n$ which are polynomial in the generators, self-consistent and compatible with the coproduct.

A recurrence procedure gives the solution

$$vn = \mathrm{e}^z\,nv + w\,(v^2-v) \qquad n\bar n = \mathrm{e}^z\,\bar n n - w\,(n+\bar n) \tag{3.9}$$

where z and w are arbitrary parameters. Evidently for vanishing z or w we recover the previous results, while we can eliminate $w\neq 0$ by rescaling the generators n and $\bar n$. It is directly verified that (3.9) together with (2.6)–(2.9) give a Hopf algebra which is a deformation of $\mathrm{Fun}(E(2))$. However, for $z\neq 0$, we can define [12]

$$m = n - \frac{w}{1-\mathrm{e}^z}(v-1) \tag{3.10}$$

and we can easily verify that v, m, $\bar m$ generate the Hopf algebra $\mathrm{Fun}(E_q(2))$. The only singularity of the transformation is the value $z=0$, where the deformation $\mathrm{Fun}(E_\ell(2))$ arises.

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